Chapter 5
Step-Index Circular Waveguide

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Introduction of Circular Waveguides

- Widely used due to low cost

Step-index Fiber

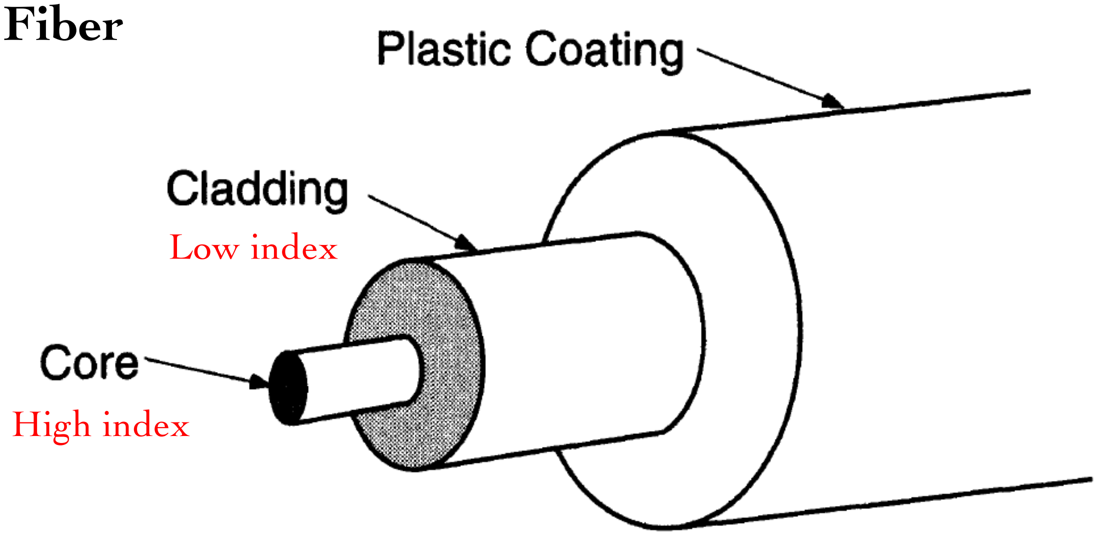


Figure 5.1 The cylindrical step waveguide consists of a high-index core surrounded by a lower-index cladding.

Wave Equations in Cylindrical Coordinates

$$\mathbf{E}(r, \phi, z) = \hat{r}E_r(r, \phi, z) + \hat{\phi}E_\phi(r, \phi, z) + \hat{z}E_z(r, \phi, z)$$

Wave equation for planar waveguide

$$\nabla^2\mathbf{E} - \mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2} = 0$$

Wave equation for circular waveguide

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla \times \nabla \times \mathbf{E} - \mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2} = 0$$

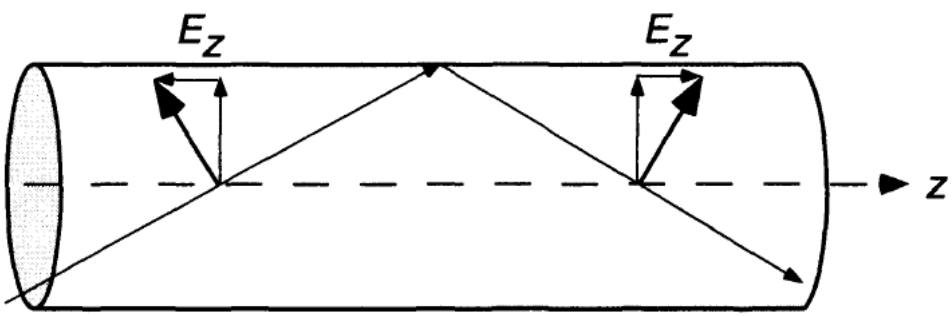


Figure 5.3 The longitudinal component of the electric field does not change through either propagation or reflection at the cylindrical surface.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \phi^2} + \frac{\partial^2 E_z}{\partial z^2} + k_0^2 n^2 E_z = 0$$

Solution of Ez

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \phi^2} + \frac{\partial^2 E_z}{\partial z^2} + k_0^2 n^2 E_z = 0$$

- Separation of variables $E_z(r, \phi, z) = R(r)\Phi(\phi)Z(z)$
- Substitute into the equation

$$R''\Phi Z + \frac{1}{r}R'\Phi Z + \frac{1}{r^2}R\Phi''Z + R\Phi Z'' + k_0^2 n^2 R\Phi Z = 0$$

- Multiple by $r^2/R\Phi Z$

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Phi''}{\Phi} + r^2\frac{Z''}{Z} + k_0^2 n^2 r^2 = 0$$

- Assume a phase term for z dependence

$$Z(z) = e^{-j\beta z}$$

β: z component of the wavevector k in the waveguide.

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Phi''}{\Phi} - r^2\beta^2 + k_0^2 n^2 r^2 = 0 \longrightarrow r^2\frac{R''}{R} + r\frac{R'}{R} - r^2\beta^2 + k_0^2 n^2 r^2 = -\frac{\Phi''}{\Phi} = \nu^2$$

Solution of Ez

$$r^2\frac{R''}{R} + r\frac{R'}{R} - r^2\beta^2 + k_0^2n^2r^2 = -\frac{\Phi''}{\Phi} = \nu^2$$

- Solve for Φ $\quad\nu$: separation constant

$$\Phi''(\phi) = -\nu^2\Phi \qquad \Phi(\phi) = Ae^{j\nu\phi} + c.c.$$

- Substitute into $r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Phi''}{\Phi} - r^2\beta^2 + k_0^2n^2r^2 = 0$

$$\rightarrow \quad r^2\frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} + r^2\left(k_0^2n^2 - \beta^2 - \frac{\nu^2}{r^2}\right)R = 0$$

Only has R(r)

- Solve using Bessel functions

$$\left(k_0^2n^2 - \beta^2 - \frac{\nu^2}{r^2}\right)$$

positive *Bessel functions of the first kind of order* ν $\quad J_\nu(\kappa r)$

$$\kappa^2 = k_0^2n^2 - \beta^2$$

negative *modified Bessel functions of the second kind of order* ν $\quad K_\nu(\gamma r)$

$$\gamma^2 = \beta^2 - k_0^2n^2$$

Bessel Functions

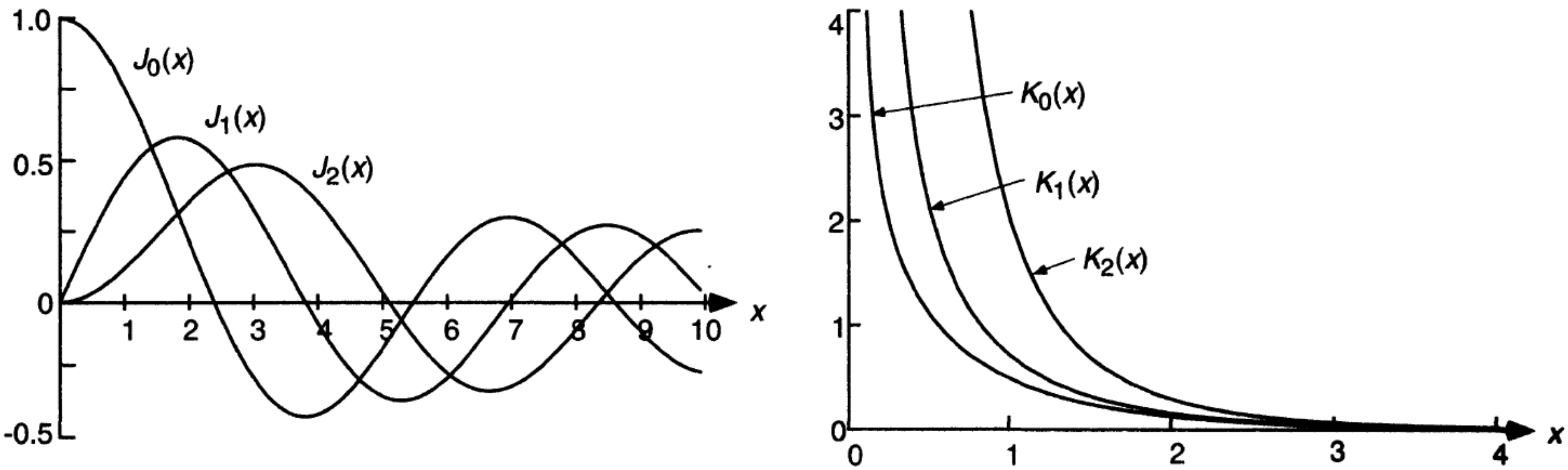


Figure 5.4 Graphs show the first three Bessel functions of the first kind, $J_\nu(\kappa r)$, and of the second kind, $K_\nu(\gamma r)$.

$$J_\nu(\kappa r) \approx \sqrt{\frac{2}{\pi \kappa r}} \cos\left(\kappa r - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \quad \text{for } \kappa r \text{ large}$$

Damped sine wave
Describe radial standing wave

$$K_\nu(\gamma r) \approx \frac{e^{-\gamma r}}{\sqrt{2\pi\gamma r}} \quad \text{for } \gamma r \text{ large}$$

Damped exponential wave
Describe evanescent wave

Field Distributions in the Step-Index Fiber

1. Oscillatory solution $J_\nu(\kappa r)$ $\kappa^2 = k_0^2 n^2 - \beta^2$

➡ $k_0 n_{core} > \beta > k_0 n_{clad}$

2. Exponential decay $K_\nu(\gamma r)$

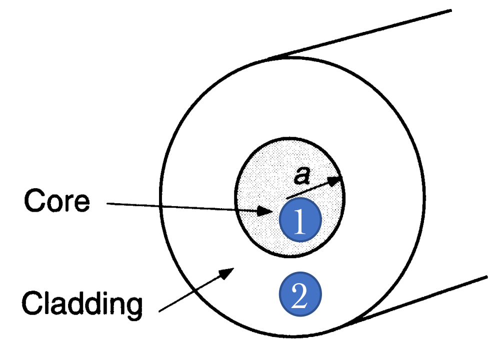


Figure 5.5 The cylindrical waveguide has a core radius of dimension a.

for $r < a$
$$E_z(r, \phi, z) = AJ_\nu(\kappa r)e^{j\nu\phi}e^{-j\beta z} + c.c.$$
$$H_z(r, \phi, z) = BJ_\nu(\kappa r)e^{j\nu\phi}e^{-j\beta z} + c.c.$$

Solve boundary conditions for A, B, C, D

for $r > a$
$$E_z(r, \phi, z) = CK_\nu(\gamma r)e^{j\nu\phi}e^{-j\beta z} + c.c.$$
$$H_z(r, \phi, z) = DK_\nu(\gamma r)e^{j\nu\phi}e^{-j\beta z} + c.c.$$

Field Distributions in the Step-Index Fiber

$$\boxed{\nabla \times \mathbf{E}} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu\frac{\partial \mathbf{H}}{\partial t} = -\mu j\omega\mathbf{H}$$

$$E_\phi = \frac{-j}{\alpha^2}\left(\frac{\beta}{r}\frac{\partial E_z}{\partial \phi} - \omega\mu\frac{\partial H_z}{\partial r}\right)$$

$$E_r = \frac{-j}{\alpha^2}\left(\frac{\mu\omega}{r}\frac{\partial H_z}{\partial \phi} + \beta\frac{\partial E_z}{\partial r}\right)$$

$$H_\phi = \frac{-j}{\alpha^2}\left(\omega\epsilon\frac{\partial E_z}{\partial r} + \frac{\beta}{r}\frac{\partial H_z}{\partial \phi}\right)$$

$$H_r = \frac{-j}{\boxed{\alpha^2}}\left(\beta\frac{\partial H_z}{\partial r} - \frac{\omega\epsilon}{r}\frac{\partial E_z}{\partial \phi}\right)$$

$$\downarrow$$

$$k_0^2 n^2 - \beta^2$$

Field Distributions in the Step-Index Fiber

Core

$$E_r = \frac{-j\beta}{\kappa^2}\left[A\kappa J_\nu'(\kappa r) + \frac{j\omega\mu\nu}{\beta r}BJ_\nu(\kappa r)\right]e^{j\nu\phi}e^{-j\beta z}$$

$$E_\phi = \frac{-j\beta}{\kappa^2}\left[\frac{j\nu}{r}AJ_\nu(\kappa r) - \frac{\omega\mu}{\beta}B\kappa J_\nu'(\kappa r)\right]e^{j\nu\phi}e^{-j\beta z}$$

$$H_r = \frac{-j\beta}{\kappa^2}\left[B\kappa J_\nu'(\kappa r) - \frac{j\omega\epsilon_{core}\nu}{\beta r}AJ_\nu(\kappa r)\right]e^{j\nu\phi}e^{-j\beta z}$$

$$H_\phi = \frac{-j\beta}{\kappa^2}\left[\frac{j\nu}{r}BJ_\nu(\kappa r) + \frac{\omega\epsilon_{core}}{\beta}A\kappa J_\nu'(\kappa r)\right]e^{j\nu\phi}e^{-j\beta z}$$

Cladding

$$E_r = \frac{j\beta}{\gamma^2}\left[C\gamma K_\nu'(\gamma r) + \frac{j\omega\mu\nu}{\beta r}DK_\nu(\gamma r)\right]e^{j\nu\phi}e^{-j\beta z}$$

$$E_\phi = \frac{j\beta}{\gamma^2}\left[\frac{j\nu}{r}CK_\nu(\gamma r) - \frac{\omega\mu}{\beta}D\gamma K_\nu'(\gamma r)\right]e^{j\nu\phi}e^{-j\beta z}$$

$$H_r = \frac{j\beta}{\gamma^2}\left[D\gamma K_\nu'(\gamma r) - \frac{j\omega\epsilon_{clad}\nu}{\beta r}CK_\nu(\gamma r)\right]e^{j\nu\phi}e^{-j\beta z}$$

$$H_\phi = \frac{j\beta}{\gamma^2}\left[\frac{j\nu}{r}DK_\nu(\gamma r) + \frac{\omega\epsilon_{clad}}{\beta}C\gamma K_\nu'(\gamma r)\right]e^{j\nu\phi}e^{-j\beta z}$$

Boundary Conditions for the Step-Index Waveguide

- Boundary condition: r=a $\quad E_z, E_\phi, H_z, \text{ and } H_\phi$ continuous

$$\begin{bmatrix} J_\nu(\kappa a) & 0 & -K_\nu(\gamma a) & 0 \\ 0 & J_\nu(\kappa a) & 0 & -K_\nu(\gamma a) \\ \frac{\beta\nu}{a\kappa^2}J_\nu(\kappa a) & j\frac{\omega\mu}{\kappa}J_\nu'(\kappa a) & \frac{\beta\nu}{a\gamma^2}K_\nu(\gamma a) & j\frac{\omega\mu}{\gamma}K_\nu'(\gamma a) \\ -j\frac{\omega\epsilon_{core}}{\kappa}J_\nu'(\kappa a) & \frac{\beta\nu}{a\kappa^2}J_\nu(\kappa a) & -j\frac{\omega\epsilon_{clad}}{\gamma}K_\nu'(\gamma a) & \frac{\beta\nu}{a\gamma^2}K_\nu(\gamma a) \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = 0$$

$$\frac{\beta^2\nu^2}{a^2}\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]^2 = \left[\frac{J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right] \left[\frac{k_0^2 n_{core}^2 J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{k_0^2 n_{clad}^2 K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right]$$

characteristic equation

oscillator $\quad J_\nu(\kappa a)$

y ν: angular mode number

m: radial mode number

Solution of A, B, C, D

$$AJ_\nu(\kappa a) = CK_\nu(\gamma a) \quad \Rightarrow \quad C = \frac{J_\nu(\kappa a)}{K_\nu(\gamma a)}A$$

$$D = \frac{J_\nu(\kappa a)}{K_\nu(\gamma a)}B$$

Continuous E_ϕ

$$B = \frac{j\nu\beta}{\omega\mu a}\left[\frac{1}{\kappa^2} + \frac{1}{\gamma^2}\right]\left[\frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)} + \frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)}\right]^{-1} A$$

Continuous H_ϕ

$$B = \frac{j\omega a}{\beta\nu}\left[\frac{n_{core}^2}{\kappa}\frac{J'_\nu(\kappa a)}{J_\nu(\kappa a)} + \frac{n_{clad}^2}{\gamma}\frac{K'_\nu(\gamma a)}{K_\nu(\gamma a)}\right]\left[\frac{1}{\kappa^2} + \frac{1}{\gamma^2}\right]^{-1} A$$

Transverse Electric and Transverse Magnetic Modes

characteristic equation $$\frac{\beta^2\nu^2}{a^2}\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]^2=\left[\frac{J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right]\left[\frac{J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right]$$

- ν: angular dependence of the solution

when $\nu = 0$ $$\left[\frac{J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right]\left[\frac{k_0^2 n_{core}^2 J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{k_0^2 n_{clad}^2 K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right]=0$$

$$B=\frac{j\nu\beta}{\omega\mu a}\left[\frac{1}{\kappa^2}+\frac{1}{\gamma^2}\right]\left[\frac{J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right]^{-1}A$$

Continuous E_ϕ

→ **A must be 0.**
Ez=0

TE mode

$$B=\frac{j\omega a}{\beta\nu}\left[\frac{n_{core}^2}{\kappa}\frac{J_\nu'(\kappa a)}{J_\nu(\kappa a)}+\frac{n_{clad}^2}{\gamma}\frac{K_\nu'(\gamma a)}{K_\nu(\gamma a)}\right]\left[\frac{1}{\kappa^2}+\frac{1}{\gamma^2}\right]^{-1}A$$

Continuous H_ϕ

→ **B=0.**
Hz=0

TM mode

Find propagation vector β

Solve $$\left[\frac{J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)} + \frac{K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right]\left[\frac{k_0^2 n_{core}^2 J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)} + \frac{k_0^2 n_{clad}^2 K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right] = 0$$

Simplified using Bessel relation $$\frac{J_\nu'}{\kappa J_\nu} = \pm\frac{J_{\nu\mp1}}{\kappa J_\nu} \mp \frac{\nu}{\kappa^2}$$

$$\frac{K_\nu'}{\gamma K_\nu} = \mp\frac{K_{\nu\pm1}}{\gamma K_\nu} \mp \frac{\nu}{\gamma^2}$$

TE: $$\left[\frac{J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)} + \frac{K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right] = 0 \quad \Longrightarrow \quad -\frac{J_1(\kappa a)}{\kappa J_0(\kappa a)} - \frac{K_1(\gamma a)}{\gamma K_0(\gamma a)} = 0$$

TM: $$\left[\frac{k_0^2 n_{core}^2 J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)} + \frac{k_0^2 n_{clad}^2 K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right] = 0$$

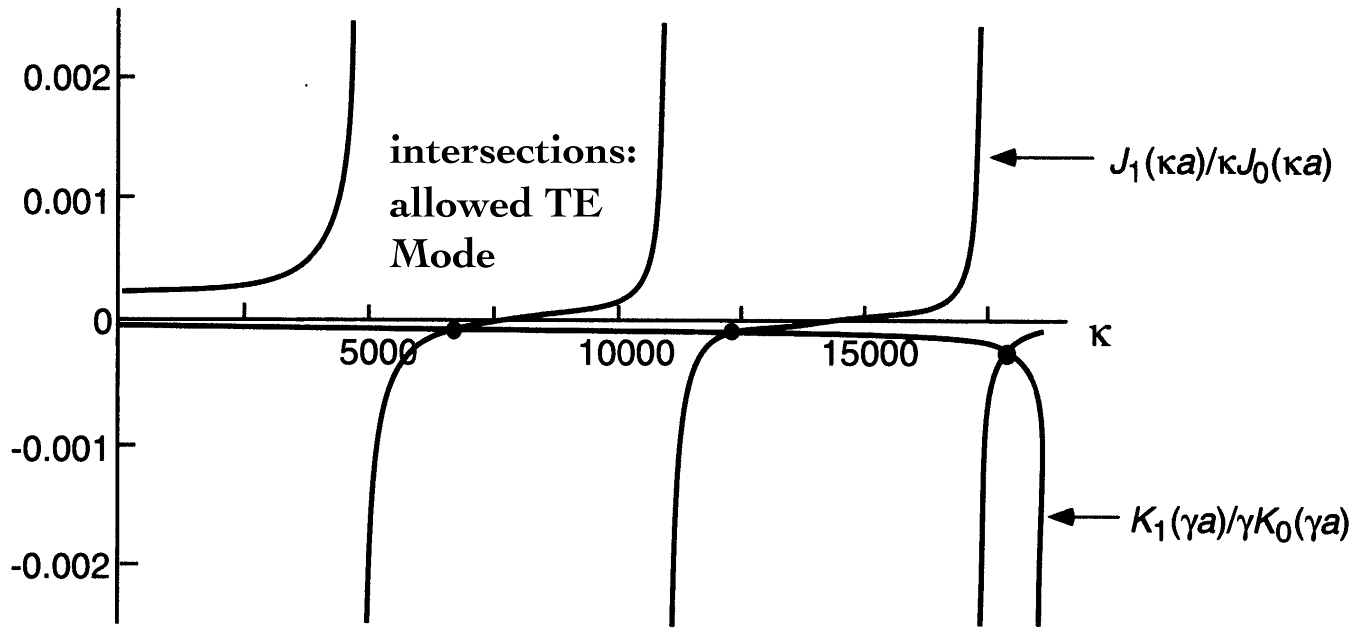
Need to solve numerically

Example of TE Mode

$$-\frac{J_1(\kappa a)}{\kappa J_0(\kappa a)} - \frac{K_1(\gamma a)}{\gamma K_0(\gamma a)} = 0$$

determine the allowed eigenvalues for β for the TE modes.

core radius $a = 5\mu\text{m}$



$$\kappa^2 = k_0^2 n^2 - \beta^2$$

Figure 5.6 The eigenvalue equation is plotted against κ for a waveguide with core index 1.5, cladding index 1.45, and wavelength 1.3μm.

Hybrid Mode

- ν≠0: both Ez and Hz

if $A = 0$	then the mode is called a	TE mode
if $B = 0$	then the mode is called a	TM mode
if $A > B$	then the mode is called an	HE mode (E_z dominates H_z)
if $A < B$	then the mode is called an	EH mode (H_z dominates E_z)

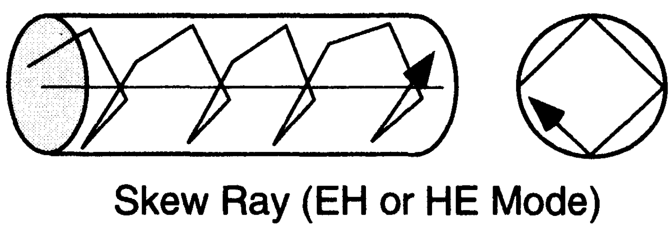


Figure 5.8 A skew ray travels in a spiral path down the fiber. The ray does not go through the origin.

Linearly Polarized Mode

$$\frac{\beta^2\nu^2}{a^2}\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]^2=\left[\frac{J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right]\left[\frac{k_0^2 n_{core}^2 J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{k_0^2 n_{clad}^2 K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right]$$

- weakly guiding approximation

$\Delta n = n_{core} - n_{clad}$ on the order of 0.001–0.005 → $n_{core} \approx n_{clad} = n$

$$\frac{\beta^2\nu^2}{a^2}\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]^2=\left[\frac{J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right]^2 k_0^2 n^2$$

if $n_{core} = n_{clad}$, then $\beta^2 = k_0^2 n^2$

using Bessel relation $\frac{J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)} = \pm\frac{J_{\nu\mp1}}{\kappa a J_\nu(\kappa a)} \mp \frac{\nu}{\kappa^2 a}$ and $\frac{K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)} = \frac{K_{\nu\pm1}(\gamma a)}{\gamma a K_\nu(\gamma a)} \mp \frac{\nu}{\gamma^2 a}$

→ $$\frac{J_{\nu\pm1}(\kappa a)}{\kappa J_\nu(\kappa a)} = \mp\frac{K_{\nu\pm1}(\gamma a)}{\gamma K_\nu(\gamma a)}$$ EH (top sign) and HE (bottom sign)

Bessel relation →

$$\kappa\frac{J_{j-1}(\kappa a)}{J_j(\kappa a)} = -\gamma\frac{K_{j-1}(\gamma a)}{K_j(\gamma a)}$$

Linearly Polarized Mode

$$\kappa \frac{J_{j-1}(\kappa a)}{J_j(\kappa a)} = -\gamma \frac{K_{j-1}(\gamma a)}{K_j(\gamma a)}$$

$j = 1$ TE, TM modes

$j = \nu + 1$ EH_ν modes

$j = \nu - 1$ HE_ν modes

Degenerate modes

TE_{0m} ⟺ TM_{0m} **Same β**

$HE_{\nu+1,m}$ ⟺ $EH_{\nu-1,m}$ **Same β**

define stable superpositions of different modes

linearly polarized primarily transverse

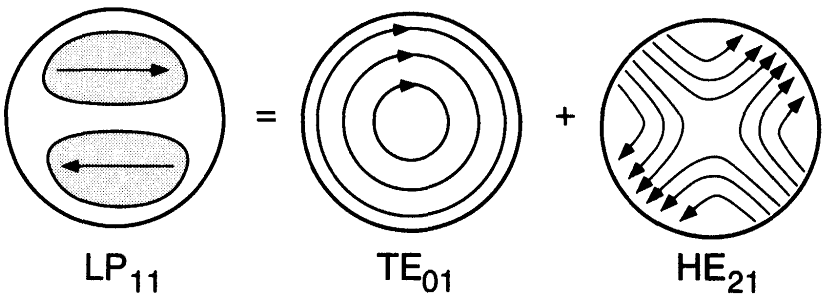


Figure 5.9 The LP_{11} mode is a superposition of the TE_{01} and HE_{21} modes. Note that the LP mode is linearly polarized, in contrast to the electric fields of the two constituent modes. E_x polarization is shown, although with appropriate superposition, an E_y polarized mode could have been created.

$LP_{1m} \rightarrow$ sum of TE_{0m}, TM_{0m}, and HE_{2m} modes

$LP_{\nu m} \rightarrow$ sum of $HE_{\nu+1,m}$ and $EH_{\nu-1,m}$ modes

$LP_{0m} \rightarrow HE_{1m}$ mode only (special case)

Normalized Frequency and Cutoff

$$\frac{\beta^2\nu^2}{a^2}\left[\frac{1}{\gamma^2}+\frac{1}{\kappa^2}\right]^2=\left[\frac{J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right]\left[\frac{k_0^2 n_{core}^2 J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{k_0^2 n_{clad}^2 K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right]$$

$J_{\nu\pm1}/\mathrm{J}_\nu \rightarrow \infty$

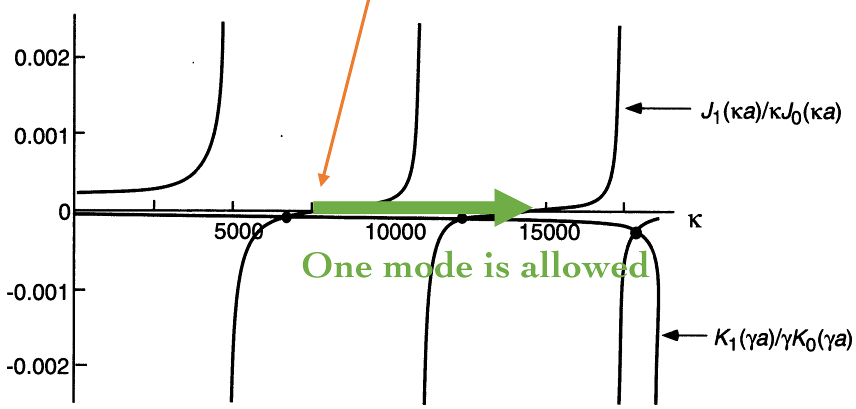


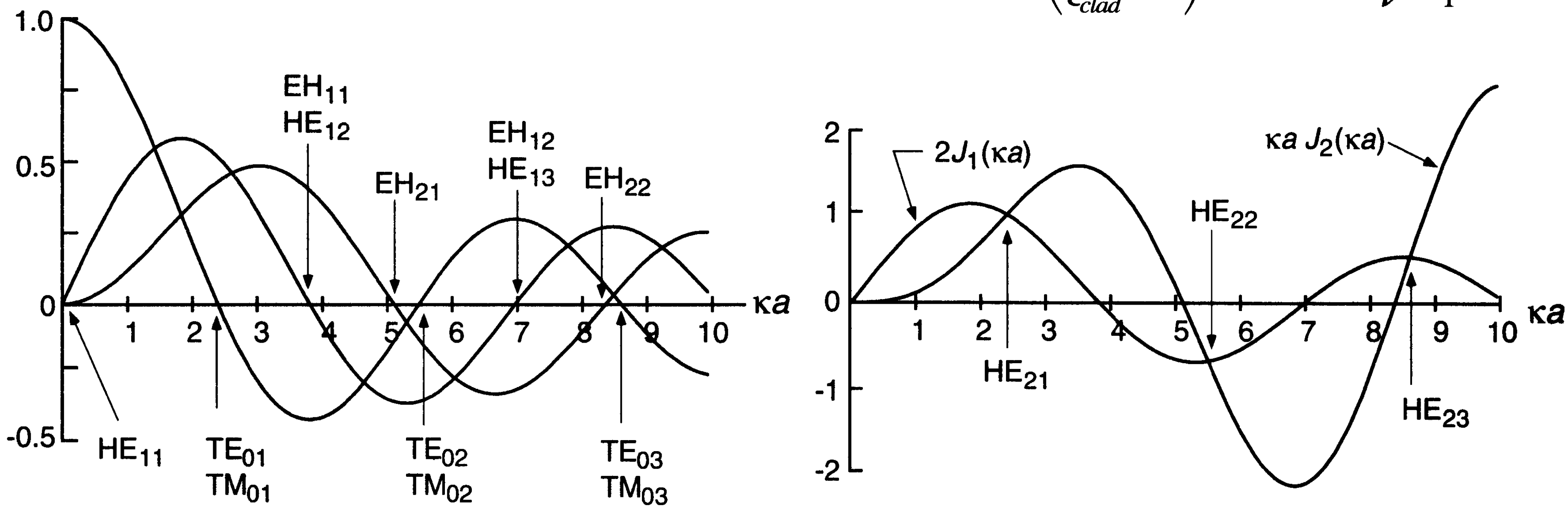
Figure 5.6 The eigenvalue equation is plotted against κ for a waveguide with core index 1.5, cladding index 1.45, and wavelength 1.3μm.

core radius $a = 5\mu$m

TE_{0m} modes	$\kappa a > m^{th}$ root of $J_0(\kappa a)$
HE_{1m} mode	$\kappa a > m^{th}$ root of $J_1(\kappa a)$
$EH_{\nu m}$ mode	$\kappa a > m^{th}$ root of $J_\nu(\kappa a)$ with the added constraint that the first root is not 0
$HE_{\nu m}$ modes	$\left(\frac{\epsilon_{core}}{\epsilon_{clad}}+1\right)J_{\nu-1}(\kappa a)=\frac{\kappa a}{\nu-1}J_\nu(\kappa a)$

Cutoff Conditions

TE_{0m} modes	$\kappa a > m^{th}$ root of $J_0(\kappa a)$
HE_{1m} mode	$\kappa a > m^{th}$ root of $J_1(\kappa a)$
$EH_{\nu m}$ mode	$\kappa a > m^{th}$ root of $J_\nu(\kappa a)$ with the added constraint that the first root is not 0
$HE_{\nu m}$ modes	$\left(\frac{\epsilon_{core}}{\epsilon_{clad}} + 1\right) J_{\nu-1}(\kappa a) = \frac{\kappa a}{\nu - 1} J_\nu(\kappa a)$



(a) **(b)**

Figure 5.11 The first three J_ν Bessel functions are plotted, with the mode cutoff conditions of a few modes indicated at the various roots of the curves. The condition $2J_1(\kappa a) = \kappa a J_2(\kappa a)$ is plotted for the HE_{2m} mode cutoff conditions. Cutoff occurs where the curves cross.

Normalized frequency (V number)

- V number: $\kappa_{max} a$

$$V \text{ number} = ak_0\sqrt{n_{core}^2 - n_{clad}^2} = \frac{2\pi a}{\lambda}\sqrt{n_{core}^2 - n_{clad}^2}$$

- Example:

Consider a step-index fiber that has a core index $n_{core} = 1.45$, a cladding index $n_{clad} = 1.44$, and a core radius of 25μm. If the excitation wavelength is 1.5μm, how many TE and TM modes will exist in the waveguide?

$$V = \frac{2\pi\, 25\mu\text{m}}{1.5\mu\text{m}}\sqrt{1.45^2 - 1.44^2}$$

$$= 17.802$$

The zeros of the $J_0(\kappa a) = 0$ occur at 2.405, 5.520, 8.654, 11.791, 14.931, 18.071

5 modes are allowed